# THE ZEL'DOVICH APPROXIMATION AND

# THE RELATIVISTIC HAMILTON-JACOBI EQUATION

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## ABSTRACT

Beginning with a relativistic action principle for the irrotational flow of collisionless matter, we compute higher order corrections to the Zel'dovich approximation by deriving a nonlinear Hamilton-Jacobi equation for the velocity potential. It is shown that the velocity of the field may always be derived from a potential which however may be a multi-valued function of the space-time coordinates. In the Newtonian limit, the results are nonlocal because one must solve the Newton-Poisson equation. By considering the Hamilton-Jacobi equation for general relativity, we set up gauge-invariant equations which respect causality. A spatial gradient expansion leads to simple and useful results which are local — they require only derivatives of the initial gravitational potential.

**Key words:** cosmology: large scale structure of the Universe — methods: analytical — galaxies: formation

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# 1.   INTRODUCTION

Recent redshift surveys of galaxies suggest that sheet-like structures are quite common in our Universe. Within a distance of $100h^{-1}$Mpc, a Great Wall appears in the Center for Astrophysics slice of the Universe, De Lapparent, Geller & Huchra (1986). A pencil beam survey, Broadhurst et al (1990) is consistent with the hypothesis that similar sheets exist out to a redshift of $z \sim \frac{1}{2}$. The Zel'dovich (1970) approximation plays a fundamental role in describing the formation of sheets that arise from the nonlinear evolution of irrotational collisionless dust (for a review, consult e.g., Shandarin & Zel'dovich (1989). In an attempt to better understand the formation of cosmic structure, we show how to compute higher order terms in the Zel'dovich approximation. We employ two principal tools: (1) a relativistic action principle for dust interacting with gravity, and (2) Hamilton-Jacobi (HJ) theory used in conjunction with a Taylor series expansion.

Dust with rest mass $m$ which flows irrotationally may be described using an action principle for a dust field $\chi$,

$$\mathcal{I}_{dust} = \int d^4x \sqrt{-g} \left\{ -\tfrac{1}{2}\rho \left( 1 + g^{\mu\nu}\chi_{,\mu}\chi_{,\nu}/m^2 \right) \right\}. \tag{1.1}$$

The four-velocity

$$U^\mu = -g^{\mu\nu}\chi_{,\nu}/m \,. \tag{1.2}$$

is proportional to the gradient of the potential $\chi$. The energy density $\rho$ is a Lagrange multiplier which ensures that the four-velocity squared is $-1$:

$$U^\mu U_\mu = -1 \,. \tag{1.3}$$

Variation with respect to $\chi$ leads to the equation of conservation

$$(\rho\chi^{,\mu})_{;\mu} = 0 \,,$$

whereas variation with respect to the 4-metric $g_{\mu\nu}$ leads to the usual stress-energy tensor:

$$T_{\mu\nu} = \rho U_\mu U_\nu \,.$$

This action principle (1.1) can be used to clarify the analysis of cosmological systems. It is relativistic and nonlinear. It will be the focal point of our discussion.

The action for gravity is the standard Einstein-Hilbert form

$$\mathcal{I}_{gravity} = \int d^4x \sqrt{-g} \, \tfrac{1}{2} \, {}^{(4)}R \,,$$

where ${}^{(4)}R$ is the Ricci scalar of the 4-metric. Here we have taken $8\pi G = 1$. We shall also assume that $m = 1$ which can always be arranged by rescaling: $\chi \to m\chi$. The action principle for dust is a special case of a general class of fluids considered by Schutz (1970, 1971); see also Salopek & Stewart (1992, 1993) who obtain exact solutions for a two-fluid system of black-body radiation and dust.

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Hamilton-Jacobi theory facilitates the generation of solutions for the corresponding field equations. In section 2, we write down the non-relativistic HJ equation for a cosmological dust field, Kofman 1991). The generating function $S$ is essentially a nonlinear velocity potential which is closely related to what is directly observable in cosmology (see e.g., Dekel et al (1990)). (We assume throughout that, as seen in a Lagrangian description, the velocity is derivable from a potential. However when caustics form $S$ may be a multivalued function of the space-time coordinates. In an Eulerian description several irrotational flows are superposed and the mean, Eulerian, flow is no longer irrotational.) The Zel'dovich approximation is the first two terms in our Taylor series expansion. Our technique clarifies the method employed by Buchert (1989, 1992, 1993) and Moutarde et al (1991). However, the higher order terms in the series are difficult to treat analytically because one must solve the Newton-Poisson equation, which is non-local.

In section 3, we bypass some of the difficulties introduced by the Newton-Poisson equation by employing the *relativistic Hamilton-Jacobi equation* which is a functional differential equation. Although the formation of structure in a cosmological setting is basically non-relativistic, the relativistic equations are useful because they maintain causality (which is not true for the Newtonian theory). The generating functional $\mathcal{S}$ is solved using a spatial gradient expansion. By choosing our time hypersurface to be a slice where the dust-field is uniform, we are able to integrate the resulting equations using an iterative technique. In this way, we solve the energy and the momentum constraints of general relativity. The energy constraint is essentially the $G^0_0$ Einstein equation; it is the relativistic generalization of the Newton-Poisson equation. The $G^0_i$ Einstein equation yields the momentum constraint which ensures that the theory is invariant under reparameterizations of the spatial coordinates ("gauge-invariance").

Our spatial gradient expansion is similar to solving Einstein's equations using a Taylor series, an approach that was initially pioneered by Lifshitz & Khalatnikov (1964). They were interested in describing the initial singularity of the big bang (see also Landau & Lifshitz (1975)). However their program was incomplete in that they did not explicitly solve the momentum constraint equation, which we solved recently under quite general conditions (Croudace, Parry, Salopek & Stewart (1994) hereafter known as CPSS; see also Salopek, Stewart, Croudace & Parry (1994)). Using the first two terms in our improved gradient expansion, we generalized the Zel'dovich approximation to a relativistic framework. We also investigated higher order terms which involve only derivatives of the initial gravitational potential — the results are local in nature. In this way, we hope to better understand the formation of sheet-like structures in the Universe.

Matarrese et al (1993) considered an alternative relativistic approach to the formation of cosmic structure. They simplified and approximated Einstein's equations by neglecting the magnetic part of the Weyl tensor. The resulting equations were local in that they did not require any information about neighboring points. They determined an exact solution and they also integrated the equations numerically. CPSS demonstrated that the exact solution of Matarrese et al (1993) corresponded to the exact Szekeres solution of Einstein's equations. They showed further that this solution was unstable. Bertschinger & Jain (1994) suggested that the outcome of this instability was cigar shaped objects which were more generic than sheets. Counter to one's intuition, they also claimed that nonlinear structures can form in void regions. However, although this approach looks promising, it is still not clear when the approximation of neglecting the magnetic part of the Weyl tensor breaks down.

In section 4, we compare the results of our gradient expansion with two exact solutions of Einstein's equations, namely planar geometries and spherical geometries (Tolman-Bondi solutions). In general, the higher order corrections to the Zel'dovich are significant. In section 5 we compare our results with the second order theory of Buchert & Ehlers (1993), and in the final section we discuss our results and state our conclusions.

## 2.    EVOLUTION OF NONLINEAR VELOCITY POTENTIAL

The Newtonian limit is essentially a non-relativistic approximation to general relativity. In special relativity the energy of a particle with rest mass $m$ is related to its momentum through

$$E = m\sqrt{1 + p^2/m^2}\,.$$

The non-relativistic limit is the first two terms in the Taylor series (in $p/m$) which yields

$$E = m + \frac{p^2}{2m}.$$

In this section, we will use a similar technique to derive the cosmological HJ equation for collisionless dust.

### 2.1    Non-relativistic Hamilton-Jacobi Equation

In a cosmological setting, the dynamics of the matter is contained in the scalar evolution equation (1.3) for the dust field:

$$\left(\dot{\chi} - N^i \chi_{,i}\right)/N = \sqrt{1 + \chi_{|i}\chi^{|i}}\,. \tag{2.1a}$$

Here we have decomposed the metric using the Arnowitt-Deser-Misner (ADM) formalism (see, e.g., Misner et al (1973)):

$$g_{00} = -N^2 + \gamma^{ij}N_i N_j, \quad g_{0i} = g_{i0} = N_i, \quad g_{ij} = \gamma_{ij}. \tag{2.1b}$$

$N$, $N_i$ and $\gamma_{ij}$ are the lapse function, the shift function, and the 3-metric, respectively. A vertical bar $|$ denotes the covariant derivative with respect to the 3-metric $\gamma_{ij}$.

We now assume that spatial gradients are small and we approximate the right-hand side of eq.(2.1a) by the first two terms of a binomial series: $1 + \frac{1}{2}\chi_{|i}\chi^{|i}$. We also work with a longitudinal gauge metric:

$$N = 1 - \Phi_H(t,x), \quad N^i = 0, \quad \gamma_{ij} = a^2(t)[1 + 2\Phi_H(t,x)]\delta_{ij}\,; \tag{2.2}$$

here $a(t) = t^{2/3}$ is the scale factor, and $\Phi_H(t,x)$ is the gauge-invariant variable of Bardeen (1980) (see also Stewart (1990)). The $x^i$ are comoving spatial coordinates. In cosmological problems which are described using longitudinal gauge, the deviations of the metric from Friedman-Robertson-Walker are of the order of $10^{-5}$, and a linear treatment of the metric

is justified (this is not true in alternative gauges such as synchronous gauge; see section 3). Next, we define the generating function $S$ through

$$\chi(t, x) = t - S(t, x) \tag{2.3}$$

which leads to the *non-relativistic Hamilton-Jacobi equation* for a particle moving in a time-dependent potential $-\Phi_H(t, x)$:

$$0 = \frac{\partial S}{\partial t} + \frac{1}{2a^2} \frac{\partial S}{\partial x^i} \frac{\partial S}{\partial x^i} - \Phi_H \ . \tag{2.4a}$$

All the terms that were dropped are negligible. The nonrelativistic HJ equation was derived earlier by Kofman (1991) who noted that it was similar to the eikonal equation in geometric optics. If for the sake of illustration the potential $\Phi_H$ is fixed, then this equation will exhibit caustics — i.e., $S$ will be multivalued. However, this does not prevent one from solving the equation (Goldstein (1981)).

The definition of the velocity potential and the four-velocity,

$$U^\mu \equiv \frac{dx^\mu}{d\tau} \equiv -g^{\mu\nu} \chi_{,\nu},$$

(where $\tau$ is proper time) implies that the coordinate velocity of a particle is related to $S$ through

$$\frac{dx^i}{dt} = \frac{1}{a^2} \frac{\partial S}{\partial x^i} \ . \tag{2.4b}$$

Hence, the generating function $S$ is essentially a velocity potential. Finally, the standard analysis of small perturbations of the metric leads to the Newton-Poisson relation

$$-\frac{2}{a^2} \nabla^2 \Phi_H = \left( \rho - \frac{4}{3\chi^2} \right) \ . \tag{2.4c}$$

The right-hand side is just the density perturbation $\delta\rho_{co}$ on a comoving slice. In general, $S$ and hence $\rho$ will be multivalued functions, in which case the right-hand is summed over all particles with the same spatial coordinate $x^i$. Finally, far within the Hubble radius, it is sufficient to approximate $\chi \sim t$ in the last equation.

It is easy to see that these equations are equivalent to the standard Newtonian equations. If we differentiate eq.(2.4a) with respect to $x^j$, one finds that

$$0 = \frac{\partial}{\partial t} \left( \frac{\partial S}{\partial x^j} \right) + \frac{dx^i}{dt} \frac{\partial^2 S}{\partial x^i \partial x^j} - \frac{\partial \Phi_H}{\partial x^j}, \tag{2.5}$$

which may be rewritten using a convective derivative,

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \frac{dx^i}{dt} \frac{\partial}{\partial x^i}, \tag{2.6}$$

leading to

$$0 = \frac{d}{dt}\left(\frac{\partial S}{\partial x^j}\right) - \frac{\partial \Phi_H}{\partial x^j} \, . \tag{2.7}$$

The physical displacement of a particle is denoted by

$$r^i = ax^i \, ,$$

and its velocity is

$$\frac{d}{dt}(ax^i) = \frac{\dot{a}}{a}r^i + a\frac{dx^i}{dt} \, .$$

In order to compute the peculiar velocity $v^i$, one subtracts the velocity, $(\dot{a}/a)r^i$, of the Hubble flow,

$$v^i = \frac{d}{dt}(ax^i) - \frac{\dot{a}}{a}r^i = \frac{1}{a}\frac{\partial S}{\partial x^i} \, .$$

Substituting in eq.(2.7), one obtains the standard cosmological equation (Peebles (1980)):

$$0 = \frac{dv^i}{dt} + \frac{\dot{a}}{a}v^i - \frac{1}{a}\frac{\partial \Phi_H}{\partial x^i} \, . \tag{2.8}$$

The density is computed from the equation of continuity which implies that

$$\rho(t, x) = \rho_0(t) \, / \det\left(\frac{\partial x^i}{\partial q^j}\right) , \tag{2.9}$$

where $q^j$ denotes the initial (Lagrangian) coordinate, and $\rho_0(t) = 4/(3t^2)$ is the energy density at early times.

It is reasonable to assume that one does not commit any gross error in assuming that the velocity is derived from a potential $S$. However, one must be aware that the velocity potential may become multivalued, and even discontinuous. It is interesting to compare our equations with the standard linear perturbation analysis of the velocity potential where one neglects the second term in eq.(2.4a) which is a quadratic perturbation. This truncated system is a poor approximation for the formation of cosmological objects because the virial theorem implies that for a bound system the kinetic energy is of the order of the potential energy.

Equations (2.4) are cumbersome to solve directly since we must solve $S$ as a function of $x^i$ which in turn is a function of the Lagrangian variables $q^j$. We will now rewrite them in a form that is more in character with the HJ formalism that we are developing. We begin by inverting the spatial variables: we assume that the Lagrangian coordinates $q^j$ are functions of the Euler coordinates $x^i$. The evolution equations for the Lagrangian coordinates are easy to write since they are constants in time: $dq^j/dt = 0$. As a result, one can write eqs.(2.4) in the form

$$0 = \frac{\partial S}{\partial t} + \frac{1}{2a^2}\frac{\partial S}{\partial x^i}\frac{\partial S}{\partial x^i} - \Phi_H \, , \tag{2.10a}$$

$$0 = \frac{\partial q^j}{\partial t} + \frac{1}{a^2}\frac{\partial S}{\partial x^i}\frac{\partial q^j}{\partial x^i} \, , \tag{2.10b}$$

$$-\frac{2}{a^2}\nabla^2\Phi_H = \frac{4}{3t^2}\left(\det\left(\frac{\partial q^j}{\partial x^i}\right) - 1\right) \, . \tag{2.10c}$$

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In this way all the dependent variables are functions of the Eulerian coordinates $x^i$.

## 2.2    Series Solution of Non-relativistic HJ Equation

We will now solve these cosmological equations by making a Taylor series expansion in the longitudinal time variable $t$:

$$S(t,x) = t\,S^{(0)}(x) + t^{5/3}\,S^{(1)}(x) + t^{7/3}\,S^{(2)}(x) + \dots , \qquad (2.11a)$$

$$q^j(t,x) = x^j + t^{2/3}\,q^{(0)j}(x) + t^{4/3}\,q^{(1)j}(x) + \dots , \qquad (2.11b)$$

$$\Phi_H(t,x) = \Phi_H^{(0)}(x) + t^{2/3}\,\Phi_H^{(1)}(x) + t^{4/3}\,\Phi_H^{(2)}(x) + \dots . \qquad (2.11c)$$

Collecting common terms we find

$$0 = S^{(0)} - \Phi_H^{(0)} , \qquad (2.12a)$$

$$0 = \frac{5}{3}S^{(1)} + \frac{1}{2}\frac{\partial S^{(0)}}{\partial x^i}\frac{\partial S^{(0)}}{\partial x^i} - \Phi_H^{(1)} , \qquad (2.12b)$$

$$0 = \frac{7}{3}S^{(2)} + \frac{\partial S^{(0)}}{\partial x^i}\frac{\partial S^{(1)}}{\partial x^i} - \Phi_H^{(2)} , \qquad (2.12c)$$

$$0 = \frac{2}{3}q^{(0)j} + \frac{\partial S^{(0)}}{\partial x^j} , \qquad (2.13a)$$

$$0 = \frac{4}{3}q^{(1)j} + \frac{\partial S^{(1)}}{\partial x^j} + \frac{\partial S^{(0)}}{\partial x^i}\frac{\partial q^{(0)j}}{\partial x^i} , \qquad (2.13b)$$

$$0 = 2q^{(2)j} + \frac{\partial S^{(2)}}{\partial x^j} + \frac{\partial S^{(1)}}{\partial x^i}\frac{\partial q^{(0)j}}{\partial x^i} + \frac{\partial S^{(0)}}{\partial x^i}\frac{\partial q^{(1)j}}{\partial x^i} , \qquad (2.13c)$$

$$\nabla^2\Phi_H^{(0)} = -\frac{2}{3}\frac{\partial q^{(0)i}}{\partial x^i} , \qquad (2.14a)$$

$$\nabla^2\Phi_H^{(1)} = -\frac{2}{3}\frac{\partial q^{(1)i}}{\partial x^i} - \frac{1}{3}\frac{\partial q^{(0)i}}{\partial x^i}\frac{\partial q^{(0)j}}{\partial x^j} + \frac{1}{3}\frac{\partial q^{(0)i}}{\partial x^j}\frac{\partial q^{(0)j}}{\partial x^i} . \qquad (2.14b)$$

The solution for the zeroth order terms is

$$S^{(0)} = \Phi_H^{(0)} = f(x), \quad q^{(0)j} = -\frac{3}{2}\frac{\partial f}{\partial x^j} . \qquad (2.15a,b)$$

The arbitrary function of the spatial coordinates, $-f(x)$, may be interpreted as the initial gravitational potential. The first order terms are expressed in terms of the *non-local* quantity $W$

$$W = \nabla^{-2}\left[\frac{\partial^2 f}{\partial x^k \partial x^l}\frac{\partial^2 f}{\partial x^k \partial x^l} - (\nabla^2 f)^2\right] , \qquad (2.16a)$$

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as

$$S^{(1)}(x) = -\frac{3}{4}\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^i} + \frac{9}{14}W \,, \tag{2.16b}$$

$$\Phi_H^{(1)}(x) = -\frac{3}{4}\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^i} + \frac{15}{14}W \,, \tag{2.16c}$$

$$q^{(1)j}(x) = \frac{\partial}{\partial x^j}\left[\frac{9}{8}\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^i} - \frac{27}{56}W\right] \,. \tag{2.16d}$$

The Zel'dovich approximation is obtained by solving for $x^i$ in favour of the Lagrangian coordinate $q^k$ using an iterative technique in eq.(2.11b):

$$x^i = q^i + \frac{3}{2}t^{2/3}\frac{\partial f}{\partial q^i} + \frac{27}{56}t^{4/3}\frac{\partial}{\partial q^i}\nabla^{-2}\left[\frac{\partial^2 f}{\partial q^k\partial q^l}\frac{\partial^2 f}{\partial q^k\partial q^l} - \left(\nabla^2 f\right)^2\right] \,. \tag{2.17}$$

Here $f$ and its derivatives are evaluated at the point $q^j$ (which is independent of time t) rather than at $x^i$ which was the case for eqs.(2.16). The first two terms are the usual Zel'dovich approximation, and the third term is essentially the improvement suggested by Buchert and Ehlers (1993). Here HJ methods simplify the derivation. For a planar gravitational system, $f(q^i) \equiv f(q^3)$, where the initial gravitational potential is a function only of the third spatial coordinate, the matrix of second derivatives for $f$ is diagonal with a single non-zero component

$$\frac{\partial^2 f}{\partial q^i\partial q^j} = \mathrm{diag}\left[0, 0, \frac{\partial^2 f}{\partial q^3\partial q^3}\right] \,, \tag{2.18}$$

and the last term in eq.(2.17) vanishes; hence the Zel'dovich approximation is essentially exact. The density is found using eq.(2.9). For a planar geometry, it is

$$\rho = \frac{4}{3t^2}\frac{1}{\left|1 + \frac{3}{2}t^{2/3}\left(\partial^2 f/\partial q^3\partial q^3\right)\right|} \,. \tag{2.19}$$

In general, further analytic progress is hampered by the non-local operator $\nabla^{-2}$ appearing in eq.(2.17).

## 3.    RELATIVISTIC HAMILTON-JACOBI THEORY

Using a general relativistic formulation, we show how to compute the higher order terms in the Zel'dovich approximation which describes cosmological collapse. We utilize the Hamilton-Jacobi equation for general relativity which is a functional differential equation that was first written by Peres (1962). We evolve the 3-metric in a spatial gradient expansion which is gauge-invariant. Our method is a significant advance over a Newtonian theory because it is local at each order. Using the improved Zel'dovich approximation, we compute the epoch of collapse.

Although the HJ formalism for general relativity can be applied to arbitrary time-hypersurfaces, in this section we will restrict ourselves to comoving surfaces where the velocity potential $\chi$ is uniform. The line element is then

$$ds^2 = -d\chi^2 + \gamma_{ij}(\chi, q)dq^i dq^j \, .$$

This choice is referred to as comoving, synchronous gauge, in which case the 3-metric $\gamma_{ij}$ has physical significance: it gives the physical distance, $ds = (\gamma_{ij}dq^i dq^j)^{1/2}$, between two comoving observers separated by an infinitesimal comoving distance $dq^i$. $\chi$ is related to the longitudinal time variable $t$ through eq.(2.3).

The *relativistic Hamilton-Jacobi equation* for the restricted generating functional $\mathcal{S}[\gamma_{ij}(x)|\chi]$ is

$$0 = \frac{\partial \mathcal{S}}{\partial \chi} + \int d^3x \left\{ \gamma^{-1/2} \frac{\delta \mathcal{S}}{\delta \gamma_{ij}(x)} \frac{\delta \mathcal{S}}{\delta \gamma_{kl}(x)} \left[ 2\gamma_{jk}(x)\gamma_{il}(x) - \gamma_{ij}(x)\gamma_{kl}(x) \right] - \frac{1}{2}\gamma^{1/2} R \right\} . \quad (3.1)$$

It is a functional differential equation. In addition one must satisfy the the momentum constraint

$$0 = -2 \left( \gamma_{ik} \frac{\delta \mathcal{S}}{\delta \gamma_{kj}(x)} \right)_{,j} + \frac{\delta \mathcal{S}}{\delta \gamma_{lk}(x)} \gamma_{lk,i}, \quad (3.2)$$

which states that the restricted generating functional $\mathcal{S}[\gamma_{ij}(x)|\chi]$ remains invariant under reparametrizations of the spatial coordinates. Eq.(3.1) is essentially the $G_0^0$ Einstein equation — it is the relativistic generalization of the Newton-Poisson equation (2.4c). Eq.(3.2) is the $G_i^0$ Einstein equation, which enforces gauge invariance. If eqs.(3.1) and (3.2) have been solved for the generating functional $\mathcal{S}$, then all of Einstein's equations are satisfied provided that the 3-metric evolves according to

$$\frac{\partial \gamma_{ij}}{\partial \chi} = 2\gamma^{-1/2} \frac{\delta \mathcal{S}}{\delta \gamma_{kl}(x)} \left( 2\gamma_{ik}\gamma_{jl} - \gamma_{ij}\gamma_{kl} \right) . \quad (3.3)$$

This is a first-order nonlinear differential equation for $\gamma_{ij}$.

### 3.1 Gradient Expansion for Generating Functional

We look for a solution of the relativistic HJ equation (3.1) in the form of a spatial gradient expansion

$$\mathcal{S} = \mathcal{S}^{(0)} + \mathcal{S}^{(2)} + \mathcal{S}^{(4)} + ..., \quad (3.4)$$

where

$$\mathcal{S}^{(0)} = -2 \int d^3x \gamma^{1/2} H(\chi) , \quad (3.5a)$$

$$\mathcal{S}^{(2)} = \int d^3x \gamma^{1/2} J(\chi) R , \quad (3.5b)$$

$$\mathcal{S}^{(4)} = \int d^3x \gamma^{1/2} \left[ L(\chi) R^2 + M(\chi) R^{ij} R_{ij} \right] . \quad (3.5c)$$

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$\mathcal{S}^{(0)}$ contains no spatial gradients whereas $\mathcal{S}^{(2)}$ contains two spatial gradients, and so on. These functionals are invariant under transformations of the spatial coordinates since they are constructed using the invariant volume $d^3x\,\gamma^{1/2}$ and the Ricci tensor $R_{kl}$ of the 3-metric $\gamma_{ij}$. $H, J, L, M$ are arbitrary functions of the velocity potential $\chi$ which are chosen so as to satisfy the relativistic HJ equation order by order:

$$H^2 + \frac{2}{3}\frac{dH}{d\chi} = 0\,, \qquad \text{(zeroth order)} \tag{3.6a}$$

$$\frac{dJ}{d\chi} + JH - \frac{1}{2} = 0\,, \qquad \text{(second order)} \tag{3.6b}$$

$$\frac{dL}{d\chi} - LH - \frac{3}{4}J^2 = 0\,, \qquad \text{(fourth order)} \tag{3.6c}$$

$$\frac{dM}{d\chi} - MH + 2J^2 = 0\,. \qquad \text{(fourth order)} \tag{3.6d}$$

As a result, the functional differential equation (3.1) has been reduced to a series of ordinary differential equations (3.6) which possess the trivial solution:

$$H = \frac{2}{3\chi}\,, \qquad \text{(zeroth order)}, \tag{3.7a}$$

$$J = \frac{3}{10}\chi\,, \qquad \text{(second order)}, \tag{3.7b}$$

$$L = \frac{81}{2800}\chi^3\,, \qquad M = -\frac{27}{350}\chi^3\,, \qquad \text{(fourth order)}. \tag{3.7c, d}$$

The constant of integration in (3.7a) can be suppressed by choosing the origin of $\chi$ appropriately. The constants of integration for (3.7b–d) become irrelevant at large $\chi$ and have been ignored. Using HJ theory, we have thus been able to solve the constraints that appear within Einstein's equations.

### 3.2 Gradient Expansion for 3-metric

We solve the evolution equation for the 3-metric by using an iterative technique. If we neglect all second order spatial gradients and replace $\mathcal{S}$ by the zeroth order approximation $\mathcal{S}^{(0)}$ in eq.(3.3) we find that the 3-metric evolves according to

$$\gamma_{ij}^{(1)}(\chi, q) = \chi^{4/3}\, k_{ij}(q)\,, \tag{3.8}$$

which is accurate to first order in spatial gradients. This is the long-wavelength approximation. The 'seed metric' $k_{ij}(q)$ is an arbitrary function that is independent of time; it describes the initial fluctuations whose wavelengths are larger than the Hubble radius. This result is very old — it was known to Lifshitz & Khalatnikov (1964).

We thus derive the higher order corrections in a systematic way. For example, accurate to third order in spatial gradients, we have found that

$$\gamma_{ij}^{(3)}(\chi, q) = \chi^{4/3} \left[ k_{ij}(q) + \frac{9}{20}\chi^{2/3} \left( \hat{R}(q)\, k_{ij}(q) - 4\hat{R}_{ij}(q) \right) \right], \quad \text{(3rd order)} \quad (3.9)$$

and to fifth order, we obtained

$$\gamma_{ij}^{(5)}(\chi, q) = \chi^{4/3} k_{ij} + \frac{9}{20}\chi^2 \left[ \hat{R}\, k_{ij} - 4\hat{R}_{ij} \right] + \qquad\qquad \text{(5th order)} \qquad (3.10)$$

$$\frac{81}{350}\chi^{8/3} \left[ k_{ij} \left( -4\hat{R}^{lm}\hat{R}_{lm} + \frac{89}{32}\hat{R}^2 + \frac{5}{8}\hat{R}_{;k}{}^{;k} \right) + \frac{5}{8}\hat{R}_{;ij} - 10\hat{R}\hat{R}_{ij} + 17\hat{R}^l{}_i\hat{R}_{lj} - \frac{5}{2}\hat{R}_{ij;k}{}^k \right].$$

Here $\hat{R}_{ij}(q)$ is the Ricci tensor of the seed metric, and a semi-colon (;) denotes its corresponding covariant derivative. (For further details, see CPSS.) Using an alternative method, these expressions have been verified by Comer et al (1994).

### 3.3 Improved Gradient Expansion for 3-metric

Unfortunately, as time increases, the third order expression, eq.(3.9), leads to nonsensical results: the determinant of the 3-metric can actually become negative. A similar problem occurs when one naively approximates a non-negative function $\cos^2(x) \sim 1 - x^2$ with the first two terms of a Taylor expansion — the approximate function falls below zero when $x > 1$. One can easily remedy this problem by expanding $\cos(x) \sim 1 - x^2/2$, and then approximating $\cos^2(x)$ by the square of this result — this technique guarantees a positive result. In an analogous way, we can improve the expansions (3.9) and (3.10) by expressing the results as a 'square.' The improved 3-third order result

$$\tilde{\gamma}_{ij}^{(3)}(\chi, q) = \chi^{4/3} \left\{ k_{il} + \frac{9}{40}\chi^{2/3} \left[ \hat{R}k_{il} - 4\hat{R}_{il} \right] \right\} k^{lm} \left\{ k_{jm} + \frac{9}{40}\chi^{2/3} \left[ \hat{R}k_{jm} - 4\hat{R}_{jm} \right] \right\},$$
$$(3.11)$$

yields the relativistic generalization of the Zel'dovich approximation (see section 3.4). The improved fifth order result is

$$\tilde{\gamma}_{ij}^{(5)}(\chi, q) = \chi^{4/3} \left\{ k_{il} + \frac{9}{40}\chi^{2/3} \left[ \hat{R}k_{il} - 4\hat{R}_{il} \right] + \chi^{4/3}F_{il} \right\} k^{lm}$$
$$\left\{ k_{jm} + \frac{9}{40}\chi^{2/3} \left[ \hat{R}k_{jm} - 4\hat{R}_{jm} \right] + \chi^{4/3}F_{jm} \right\}, \qquad (3.12a)$$

where

$$F_{ij} = \frac{81}{5600} \left\{ k_{ij} \left[ -32\hat{R}^{lm}\hat{R}_{lm} + 5\hat{R}_{;k}{}^{;k} + \frac{41}{2}\hat{R}^2 \right] - 66\hat{R}\hat{R}_{ij} + 108\hat{R}^l{}_i\hat{R}_{lj} - 20\hat{R}_{ij;k}{}^{;k} + 5\hat{R}_{;ij} \right\}.$$
$$(3.12b)$$

In comoving synchronous gauge, conservation of particle number implies that $\rho\gamma^{1/2}$ is independent of time, and hence the number density $\rho$ is given by

$$\rho(\chi, q) = \frac{4}{3} \, k^{1/2}(q) \, \gamma^{-1/2}(\chi, q) \, . \tag{3.13}$$

More details are given in Salopek & Stewart (1992), CPSS and Parry et al (1994).

The argument for using the 'square' trick is heuristic, and it can placed on firmer theoretical grounds by showing that it leads to an exact solution of Einstein's equations. Our gradient expansion solution is essentially a Taylor series that is accurate for small $\chi$. *A priori* there is no reason that it should work well in the regime where pancake structures are forming. However, after choosing an appropriate seed, the expressions for the 'square' agree precisely with the exact Szekeres (1975) solution which describes axisymmetric pancake formation (see CPSS). Hence, we have essentially *fitted* the late-time behavior to the exact solution. This method is similar to the application of WKB approximation in quantum mechanics. There, one matches two WKB solutions across the classical turn-around point by using the exact Airy solution for a linear potential.

### 3.4 Modeling the Observable Universe

Further simplifications arise when we wish to model our observable Universe. Although gravitational radiation arising from inflation (Salopek 1992) can indeed contribute significantly to the cosmic microwave background anisotropy observed by the COBE satellite, its interaction with cold-dark-matter is negligible during the matter-dominated era. For the purposes of describing the formation of galaxies, it is adequate to write the initial seed metric in the isotropic form:

$$k_{ij}(q) = \delta_{ij} \exp\left(\frac{10}{3} f(q)\right) \, . \tag{3.14}$$

The arbitrary function $f(q)$ is Bardeen's gauge-invariant variable at early times, and it can be interpreted as the negative of the initial gravitational potential. Whereas all of the expressions in section 3.3 were invariant under the group of spatial coordinate transformations, eq.(3.14) is invariant only under the subgroup of spatial rotations. We have thus sacrificed some of the gauge-invariance in favor of expressing the seed-metric in a very simple form involving the single function $f(q)$. (However, in the derivation of eqs.(3.11), (3.12), it is very useful to assume that all quantities are fully gauge-invariant, otherwise the expressions become unwieldy.) One may readily compute the Ricci tensor for the above seed metric:

$$\hat{R}_{ij}(q) = \frac{5}{3} \left(-f_{,ij} - \delta_{ij} f^{,l}_{,l}\right) + \frac{25}{9} \left(f_{,i} f_{,j} - f_{,l} f^{,l} \delta_{ij}\right) \, . \tag{3.15}$$

Our results may be simplified even further if we make one additional approximation: at each order in the gradient expansion, we will retain only the lowest order terms in $f$ since $f \sim 10^{-5}$ is much smaller than unity. For example, we obtain

$$\gamma_{ij}^{(1)}(\chi, q) = \chi^{4/3} \delta_{ij}, \quad \text{(first)} \tag{3.16}$$

at first order,

$$\tilde{\gamma}_{ij}^{(3)}(\chi, q) = \chi^{4/3} \left[ \delta_{il} + \frac{3}{2} \chi^{2/3} f_{,il} \right] \left[ \delta_j^l + \frac{3}{2} \chi^{2/3} f^{,l}{}_{,j} \right] , \quad \text{(improved 3rd)} \tag{3.17}$$

at improved third order, and

$$\tilde{\gamma}_{ij}^{(5)}(\chi, q) = \chi^{4/3} \left[ \delta_{il} + \frac{3}{2} \chi^{2/3} f_{,il} + \chi^{4/3} F_{il} \right] \left[ \delta_j^l + \frac{3}{2} \chi^{2/3} f^{,l}{}_{,j} + \chi^{4/3} F^l{}_j \right] \quad \text{(improved 5th)} \tag{3.18a}$$

at improved fifth order where

$$F_{ij} = \frac{27}{56} \left[ 4 \left( f_{,il} f^{,l}{}_{,j} - f^{,l}{}_{,l} f_{,ij} \right) + \delta_{ij} \left( (f^{,l}{}_{,l})^2 - f^{,lm} f_{,lm} \right) \right] . \tag{3.18b}$$

It is important to note that the number density at the improved third order is exactly the result quoted by Zel'dovich:

$$\tilde{\rho}^{(3)}(\chi, q) = \frac{4}{3\chi^2} \ / \det \left[ \delta^i{}_l + \frac{3}{2} \chi^{2/3} f^{,i}{}_{,l}(q) \right] . \tag{3.19}$$

Hence we are justified in claiming that our improved third order spatial gradient approximation reproduces the usual Zel'dovich approximation (CPSS, Salopek et al (1994)).

### 3.5 Calculation of Collapse Epoch

In order to understand the consequences of the improved fifth order results, we will compute the time when the density becomes infinite.

For a given spatial point, we will rotate our coordinates so that the matrix of second derivatives is diagonal,

$$f_{,ij} \equiv \frac{\partial^2 f}{\partial q^i \partial q^j} = \text{diag}[-d_1, -d_2, -d_3] , \tag{3.20}$$

Fortunately, the 3-metric at improved fifth order, eq.(3.18), is also diagonal, which gives it a huge advantage over the original form eq.(3.12). The density $\tilde{\rho}^{(5)}$ at improved fifth order becomes infinite when

$$\det \left[ \delta_{ij} + \frac{3}{2} \chi^{2/3} f_{,ij} + \frac{27}{56} \chi^{4/3} \left[ 4 \left( f_{,il} f^{,l}{}_{,j} - f^{,l}{}_{,l} f_{,ij} \right) + \delta_{ij} \left( (f^{,l}{}_{,l})^2 - f^{,lm} f_{,lm} \right) \right] \right] = 0. \tag{3.21}$$

If $d_1$ is the largest positive eigenvalue of $-f_{,ij}$, then this occurs when the scale factor $(a \equiv \chi^{2/3})$ is given by

$$a_{coll}^{(5)} = \frac{4}{3d_1} \ / \left[ 1 + \sqrt{1 + \frac{12}{7} \left( \frac{d_3}{d_1} + \frac{d_2}{d_1} - \frac{d_2}{d_1} \frac{d_3}{d_1} \right)} \right] , \tag{3.22}$$

which is a function of $q$. In Fig. 1 we plot the collapse time for various values of the order pair $(d_2/d_1, d_3/d_1)$ which describe various ellipsoid configurations of the quadratic form $f_{,ij}\Delta q^i \Delta q^j = 1$. (1,1) is a sphere, (1,0) and (0,1) are cylinders, and the line $d_2/d_1 = d_3/d_1$ represents a class of oblate spheroids.

The origin (0,0) corresponds to the situation when $d_1 >> |d_2|, |d_3|$ which describes a sheet or plane. As expected from the discussion at the end of section 2.2, the standard Zel'dovich approximation (improved 3rd order) is essentially exact for this case. However, when all the eigenvalues are equal, $d_1 = d_2 = d_3$, describing a locally spherical system, then $a_{coll}^{(5)}/a_{coll}^{(3)} = 0.755$ represents a 24% decrease over the improved third order result. The fifth order correction can be quite significant although it probably breaks down before one reaches $a_{coll}^{(5)}/a_{coll}^{(3)} = 2$ shown in Fig. 1.

# 4.  COMPARISON WITH EXACT SOLUTIONS OF EINSTEIN'S EQUATIONS

In CPSS, we compared our improved spatial gradient expansion with the Szekeres solution. We will now compare with two well-known exact solutions: planar and spherical geometries.

## 4.1  Planar Geometries and the Zel'dovich Approximation

For a planar gravitational system, the Zel'dovich approximation is essentially exact until caustic formation provided that the *initial* gravitational potential $f(z)$ satisfies,

$$f(z) \ll 1, \tag{4.1a}$$

and

$$\left(\frac{\partial f}{\partial z}\right)^2 \ll \frac{\partial^2 f}{\partial z^2}, \tag{4.1b}$$

where $z$ denotes the spatial coordinate. This result is well-known for the Newtonian theory (see, *i.e.*, Kofman (1991)) but here we will use the methods of general relativity. For a perturbation $f(z) = f_0 \cos(kz)$, with $f_0 < 10^{-5}$ consistent with observations, condition (4.1a) implies condition (4.1b), but to be precise and mathematically rigorous, we will assume that both conditions hold.

The metric for a planar gravitational system was given in the following form by Eardley, Liang & Sachs (1972):

$$ds^2 = -d\chi^2 + \phi^2(\chi, z)\left(dx^2 + dy^2\right) + \psi^2(\chi, z)dz^2 \tag{4.2a}$$

where

$$\left(\frac{d\phi}{d\chi}\right)^2 = \frac{\lambda(z)}{\phi} + \kappa^2(z), \tag{4.2b}$$

and

$$\psi = \frac{\partial \phi}{\partial z}/\kappa(z). \tag{4.2c}$$

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Here $\lambda \equiv \lambda(z)$, $\kappa \equiv \kappa(z)$ are independent of time.

If $\lambda > 0$ (which is the case of physical interest — see below), then the exact solution is given parametrically in terms of $\eta$,

$$\chi = \frac{\lambda}{2\kappa^3} (\sinh\eta - \eta) , \qquad (4.3a)$$

$$\phi = \frac{\lambda}{2\kappa^2} (\cosh\eta - 1) , \qquad (4.3b)$$

$$\psi = \frac{\lambda}{2\kappa^3} \left[ \left( \frac{\lambda'}{\lambda} - 2\frac{\kappa'}{\kappa} \right) (\cosh\eta - 1) + \left( -\frac{\lambda'}{\lambda} + 3\frac{\kappa'}{\kappa} \right) \sinh\eta \frac{(\sinh\eta - \eta)}{(\cosh\eta - 1)} \right] , \qquad (4.3c)$$

where prime denotes a derivative with respect to $z$: $' = \partial/\partial z$. It is useful to let

$$\lambda = \frac{4}{9}\exp(5f) , \qquad \kappa = \frac{5f'}{3}, \qquad (4.4)$$

because at early times (small $\chi$), one recovers the standard long-wavelength initial conditions,

$$\phi = \psi = \chi^{2/3} \exp\left( \frac{5f}{3} \right) \qquad \text{(early times)}, \qquad (4.5)$$

which describes an initially isotropic metric. The metric variables become

$$\chi = \frac{6}{125} \frac{\exp(5f)}{(f')^3} (\sinh\eta - \eta) , \qquad (4.6a)$$

$$\phi = \frac{2}{25} \frac{\exp(5f)}{(f')^2} (\cosh\eta - 1) , \qquad (4.6b)$$

$$\psi = \frac{6}{125} \frac{\exp(5f)}{(f')^3} \left[ \left( 5f' - 2\frac{f''}{f'} \right) (\cosh\eta - 1) + \left( -5f' + 3\frac{f''}{f'} \right) \sinh\eta \frac{(\sinh\eta - \eta)}{(\cosh\eta - 1)} \right] . \qquad (4.6c)$$

Following Tomita (1975), we expand our results in a Taylor series in $\chi^{2/3}$:

$$\eta = 5\chi^{1/3} f' \exp\left( -\frac{5f}{3} \right) \left[ 1 - \frac{5}{12}\chi^{2/3} (f')^2 \exp\left( -\frac{10f}{3} \right) + \right.$$

$$\left. \frac{25}{56}\chi^{4/3} (f')^4 \exp\left( -\frac{20f}{3} \right) + \dots \right], \qquad (4.7)$$

$$\phi = \chi^{2/3}\exp\left( \frac{5f}{3} \right) \left[ 1 + \frac{5}{4}\chi^{2/3} (f')^2 \exp\left( -\frac{10f}{3} \right) - \frac{75}{112}\chi^{4/3} (f')^4 \exp\left( -\frac{20f}{3} \right) \dots \right], \qquad (4.8)$$

$$\psi = \chi^{2/3}\exp\left(\frac{5f}{3}\right)\left[1 + \left(\frac{3}{2}f'' - \frac{5}{4}(f')^2\right)\chi^{2/3}\exp\left(-\frac{10f}{3}\right) +$$

$$\left(-\frac{45}{28}(f')^2 f'' + \frac{225}{112}(f')^4\right)\chi^{4/3}\exp\left(-\frac{20f}{3}\right) + \dots\right]. \tag{4.9}$$

It can be shown that eqs.(4.8) and (4.9) are in exact agreement with the improved spatial gradient expansion given in eq.(3.12).

Let's now focus our attention on the last equation, and apply the approximations of eq.(4.1) which leads to:

$$\psi = \chi^{2/3}\left[1 + \frac{3}{2}f''\chi^{2/3} - \frac{45}{28}(f')^2 f''\chi^{4/3} + \dots\right]. \tag{4.10}$$

When caustic formation occurs, the second term is $-1$ and the third term is negligible provided eq.(4.1b) is valid. (All higher order terms are negligible as well.) To an excellent approximation, the fields evolve according to

$$\eta = 5f'\chi^{1/3}, \tag{4.11}$$

$$\phi = \chi^{2/3}, \tag{4.12}$$

$$\psi = \chi^{2/3}\left(1 + \frac{3}{2}\chi^{2/3}f''\right), \tag{4.13}$$

before caustic formation. The last expression is the Zel'dovich approximation because the corresponding density is

$$\rho = \frac{4}{3\chi^2}\frac{1}{\left|1 + \frac{3}{2}\chi^{2/3}f''\right|}. \tag{4.14}$$

Hence in a planar system, the Zel'dovich approximation is essentially exact until caustics form. Additional boundary conditions are needed to describe what happens later. Two possibilities are:

(1) The particles are collisionless and they pass through each other. (These are cold-dark-matter particles; they comprise approximately 95% of the mass density of the Universe.)

(2) The particles stick together on collision, forming a single particle whose velocity is given by momentum conservation. This presumably describes the baryon component (which comprises about 5% of the mass density of the Universe.)

We will not discuss the issue of caustics further in this paper.

## 4.2    Spherical Systems and the Improved Zel'dovich Approximation

For spherical systems, the usual Zel'dovich approximation may not be very accurate. As a result, we investigate higher order corrections.

The Tolman-Bondi metric is

$$ds^2 = -d\chi^2 + r^2(\chi, q)\left(d\theta^2 + \sin^2\theta d\phi^2\right) + \psi^2(\chi, q)dq^2, \qquad (4.15)$$

where

$$\left(\frac{dr}{d\chi}\right)^2 = \frac{2M}{r} + \left[\Gamma^2(q) - 1\right], \qquad (4.16a)$$

$$\psi = \frac{\partial r}{\partial q}/\Gamma(q), \qquad (4.16b)$$

and $M \equiv M(q)$, $\Gamma \equiv \Gamma(q)$ are functions of the comoving radial coordinate $q$; they are independent of time $\chi$. (See Misner et al (1973).)

We now choose

$$M = \frac{2}{9}q^3\exp\left(5f(q)\right),$$

$$\Gamma = 1 + \frac{5}{3}q\frac{\partial f}{\partial q},$$

so that for small $\chi$, the metric variables evolve according to

$$r/q = \psi = \chi^{2/3}\exp\left(\frac{5f}{3}\right), \qquad (4.17)$$

which yields an isotropic metric at early times.

For $\Gamma^2 - 1 > 0$, the exact solution is analogous to the planar case

$$\chi = \frac{2}{9}\frac{\exp\left(5f\right)q^3}{\left(\Gamma^2 - 1\right)^{3/2}}\left(\sinh\eta - \eta\right), \qquad (4.18a)$$

$$r = \frac{2}{9}\frac{\exp\left(5f\right)q^3}{\left(\Gamma^2 - 1\right)}\left(\cosh\eta - 1\right). \qquad (4.18b)$$

If $\Gamma^2 - 1 < 0$, then the above formulae are still valid; in this case $\sqrt{\Gamma^2 - 1}$ and $\eta$ are pure imaginary.

The planar case may be recovered if one identifies $\phi = r/q$, and then assumes

$$q\frac{\partial f}{\partial q} \gg 1; \qquad (4.19)$$

crudely speaking, this is the large radius limit. However, we have chosen to treat the spherically symmetric case separately because the results are qualitatively different.

Once again, we expand in a Taylor series in $\chi^{2/3}$

$$r = \chi^{2/3} q \exp\left(\frac{5f}{3}\right) \left\{ 1 + \frac{5}{4}\chi^{2/3}\left[\left(\frac{\partial f}{\partial q}\right)^2 + \frac{6}{5q}\frac{\partial f}{\partial q}\right] \exp\left(-\frac{10f}{3}\right) \right.$$

$$- \frac{75}{112}\chi^{4/3}\left[\left(\frac{\partial f}{\partial q}\right)^2 + \frac{6}{5q}\frac{\partial f}{\partial q}\right]^2 \exp\left(-\frac{20f}{3}\right)$$

$$\left. + \frac{2875}{4032}\chi^2\left[\left(\frac{\partial f}{\partial q}\right)^2 + \frac{6}{5q}\frac{\partial f}{\partial q}\right]^3 \exp\left(-10f\right) + \ldots \right\}, \qquad (4.20a)$$

$$\psi = \chi^{2/3}\exp\left(\frac{5f}{3}\right) \left\{ 1 + \chi^{2/3}\left[\frac{3}{2}\frac{\partial^2 f}{\partial q^2} - \frac{5}{4}\left(\frac{\partial f}{\partial q}\right)^2\right] \exp\left(-\frac{10f}{3}\right) \right. \qquad (4.20b)$$

$$- \frac{45}{28}\chi^{4/3}\left[\left(\frac{\partial f}{\partial q}\right)^2 + \frac{6}{5q}\frac{\partial f}{\partial q}\right]\left[\frac{\partial^2 f}{\partial q^2} - \frac{1}{2q}\frac{\partial f}{\partial q} - \frac{5}{4}\left(\frac{\partial f}{\partial q}\right)^2\right] \exp\left(-\frac{20f}{3}\right)$$

$$\left. + \frac{575}{224}\chi^2\left[\left(\frac{\partial f}{\partial q}\right)^2 + \frac{6}{5q}\frac{\partial f}{\partial q}\right]^2\left[\frac{\partial^2 f}{\partial q^2} - \frac{2}{3q}\frac{\partial f}{\partial q} - \frac{25}{18}\left(\frac{\partial f}{\partial q}\right)^2\right] \exp\left(-10f\right) + \ldots \right\}.$$

(Of course, eqs.(4.20) are in exact agreement with the improved spatial gradient expansion eq.(3.12).) In general, this series does not truncate after the first two terms. For example, consider $f(q) = f_0 \cos(Aq)$ at $q = 0$.

In Fig. 2a, for $\Gamma^2 - 1 < 0$, we show the exact solution for the radius $r(\chi, q)$ (bold line) with $q = 0$. This case describes the collapse of a high density region which is spherically symmetric. The light line is the first term (long-wavelength approximation). Also shown are the terms of order 3, 5 and 7 in spatial gradients. It is encouraging that the series appears to be converging before total collapse occurs. In Fig. 2b we consider $\Gamma^2 - 1 > 0$ describing the evolution of a void region. Here, the series does not converge for $\chi > 1.4$ (which is also the radius of convergence of Fig. 2a). Even then the higher order terms are relatively close to the exact result.

## 5.   COMPARISON WITH THE BUCHERT-EHLERS SCHEME

Various schemes offering higher order improvements to the Zel'dovich approximation within Newtonian theory have been suggested, including Buchert (1989, 1992, 1993), Buchert & Ehlers (1993), Moutarde et al (1991), Gramann (1993), among others. Naturally one would like to be able to compare such theories with the one presented here. We have concentrated on the Buchert-Ehlers second order scheme (which we derived in section 2), because their theory is clearly and unambiguously expressed. Although there is a related third order scheme (Buchert 1993), the calculations required for a comparison are too unwieldy to discuss here.

All Newtonian theories are non-local because at some stage the Newton-Poisson equation has to be solved. We therefore discuss only the restricted model described in section 5 of their paper which depends on the scalar function $f(q)$ which obeys the constraint

$$\epsilon_{ijk} f_{,j} f_{,kmm} = 0 \,. \tag{5.1}$$

For such a choice of $f(q)$, the non-local Newtonian expression (2.17) becomes local:

$$x^i_{(B)} = q^i + \tfrac{3}{2} t^{2/3} f_{,i} + \tfrac{27}{56} t^{4/3} \left( f_{,ik} f_{,k} - f_{,kk} f_{,i} \right) \,. \tag{5.2}$$

Working in synchronous gauge their approximation implies a spatial 3-metric

$$
\begin{aligned}
\gamma_{(B)ij} =& t^{4/3} \frac{\partial x_{(B)l}}{\partial q^i} \, \frac{\partial x^l_{(B)}}{\partial q^j} \\
=& t^{4/3} \left( \delta_{il} + \tfrac{3}{2} t^{2/3} f_{,il} + t^{4/3} \mathcal{F}_{il} \right) \left( \delta^l{}_j + \tfrac{3}{2} t^{2/3} f^{,l}{}_{,j} + t^{4/3} \mathcal{F}^l{}_j \right),
\end{aligned} \tag{5.3}
$$

where

$$\mathcal{F}_{ij} = \tfrac{27}{56} \left( f_{,ijk} f^{,k} + f_{,ik} f_{,j}{}^k - f^{,k}{}_{,(i} f_{,j)k} - f^{,k}{}_{,k} f_{,ij} \right) \,. \tag{5.4}$$

If we set $t = \chi$ and ignore $\mathcal{F}_{ij}$, then their resulting first order theory agrees perfectly with our improved third order theory, eq.(3.17). Both theories imply the standard Zel'dovich approximation expression for the density, eq.(3.19). However their second order theory ($\mathcal{F}_{ij} \neq 0$) does not in general agree with our improved fifth order approximation, eq.(3.18). Their $\mathcal{F}_{ij}$ is not the same as our $F_{ij}$, eq.(3.18b). In particular the geodesics, which relate Eulerian to Lagrangian positions as a function of time, will in general differ.

For spherical geometries, the two expressions, $\mathcal{F}_{ij}$ and $F_{ij}$, are indeed consistent. Moreover, it is straightforward to verify in general, that their traces coincide: $\mathcal{F}^k{}_k = F^k{}_k$. This implies that $\det \gamma_{(B)ij} = \det \tilde{\gamma}^{(5)}_{ij}$: at this order, both estimates of the density as a function of $t$ and $q$ will agree. The discrepancy between $\mathcal{F}_{ij}$ and $F_{ij}$ is still under investigation.

## 6. DISCUSSION AND CONCLUSIONS

Since the velocity potential plays such a prominent role in the Zel'dovich approximation, we began our analysis with a relativistic action principle for a dust field $\chi$. In the Newtonian limit, we used the *non-relativistic HJ equation* to solve for the velocity potential which is expressed in terms of a Taylor series. We recovered the results of Buchert & Ehlers (1993). Unfortunately the final expressions for the density are nonlocal, which make further analytic progress difficult.

In conflict with the principle of causality, nonlocal terms suggest that information may be transferred faster than the speed of light. By considering a general relativistic formulation, we avoid this problem. We solved Einstein's equations using a spatial gradient approximation. We derived the Zel'dovich approximation using the first two terms in our expansion. We also investigated higher order corrections. Consistent with causality, our expressions are local in that they involve only derivatives of the initial gravitational

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potential. The simplicity of our final results suggest that they will be a useful tool in cosmology. For arbitrary initial conditions, we computed the epoch of collapse. We hope to apply our techniques to the interpretation of redshift surveys.

As a check, we have compared our gradient expansion with two exact solutions of Einstein's equations: planar and spherical geometries. Using a Taylor series in the planar case, it is straightforward to show that the first two terms in the improved expansion are sufficient — all the higher order terms are very small. Hence in a general relativistic formulation, we rederive the fact that the Zel'dovich approximation is essentially exact for planar geometries (for a Newtonian derivation, see, e.g., Kofman (1991)). As was expected from the Newtonian theory, the results for spherical geometries differ quantitatively from the planar case. Near zero radius, the series does not truncate for a spherical system, and the higher corrections to the Zel'dovich approximation are indeed significant. (However, in the large radius limit the spherical case reduces to the planar one.) Our general relativistic approach, of, course is valid in general — not just for spherical and planar geometries.

Comparison with Newtonian theories, e.g., that of Buchert and Ehlers (1993), is difficult because they are non-local. For the special case of spherical geometries, our results are completely consistent. However, in general we have not been able to show that two methods are equivalent. In any case, our relativistic formalism does lead to simpler expressions for the final results (see eq.(3.18)).

## ACKNOWLEDMENTS

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# Figure Captions

**Figure 1.** Higher order corrections to the standard Zel'dovich approximation are in general significant. For the improved fifth order calculation, the scale factor, $a_{coll}^{(5)} \equiv \chi^{2/3}$, at the time of collapse is shown as a function of the eigenvalues $d_i$ of $-f_{,ij}$ where $-f$ is the initial gravitational potential. From the bottom to the top, the curves represent $a_{coll}^{(5)}/a_{coll}^{(3)} = 2.0, 1.2, 1.0$ (bold), $0.9, 0.8$ and $0.755$ (sharp corner), where the first and last are the maximum and minimum values, respectively. $d_1$ is the largest positive eigenvalue and $a_{coll}^{(3)} = 2/(3d_1)$ is the standard Zel'dovich approximation result.

**Figure 2.** We compare our gradient expansion (1st, 3rd, 5th and 7th order) with the exact Tolman-Bondi solution (bold curve) for a spherical system. In (a), spherical collapse is shown. Apparently, the spatial gradient expansion is converging. In (b), the radius $r$ expands indefinitely. The radius of convergence $\chi_{crit} = 1.4$ is finite and it coincides with the collapse time given in (a). Even beyond the radius of convergence, the successive approximations do not differ much from the exact solution.

This figure "fig1-1.png" is available in "png" format from:

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COLLAPSE EPOCH FOR IMPROVED ZEL'DOVICH APPROXIMATION
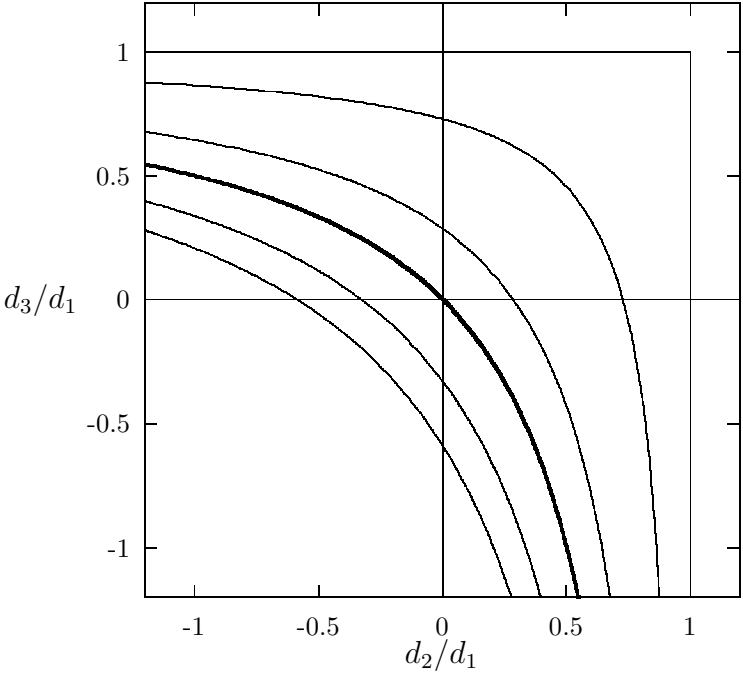
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SPHERICAL COLLAPSE

SPHERICAL EXPANSION